Self-propulsion of V-shape micro-robot

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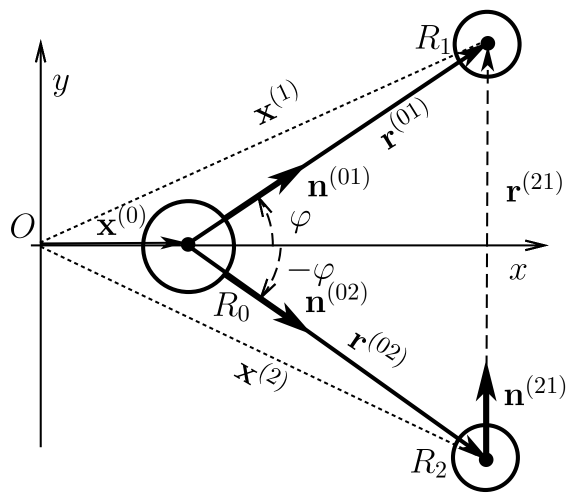
In this paper we study the self-propulsion of a symmetric V-shape micro-robot (or V-robot) which consists of three spheres connected by two arms with an angle between them; the arms' lengths and the angle are changing periodically. Using an asymptotic procedure containing two-timing method and a distinguished limit, we obtain analytic expressions for the self-propulsion velocity and Lighthill's efficiency. The calculations show that a version of V-robot, aligned perpendicularly to the direction of self-swimming, is both the fastest one and the most efficient one. We have also shown that such V-robot is faster and more efficient than a linear three-sphere micro-robot. At the same time the maximal self-propulsion velocity of V-robots is significantly smaller than that of comparable microorganisms.

1. Introduction and formulation of problem

The studies of micro-robots represent a flourishing modern research topic which strives to create a fundamental base for modern applications in medicine and technology, see *e.g.* Purcell (1977), Becker *et.al.* (2003), Najafi & Golestanian (2004), Dreyfus *et.al.* (2005), Earl *et.al.* (2007), Chang *et.al.* (2007), Alouges *et.al.* (2008), Gilbert *at.al.* (2010), Golestanian & Ajdari (2008), Golestanian & Ajdari (2009), Alexander *et.al.* (2009), Leoni *et.al.* (2009), Lauga (2011). The simplicity of both time-dependence and geometry of micro-robots represents their major advantage in contrast with extreme complexity of self-swimming microorganisms, *e.g.* Pedley & Kessler (1987), Vladimirov *et.al.* (2004), Pedley (2009), Polin *et.al.* (2009). This advantage allows to describe the motion of micro-robots in a greater depth. In this paper, we study the self-propulsion of a symmetric V-shape three-sphere micro-robot (for brevity we call it V-robot), see the figure. We employ two-timing method and distinguished limit arguments, which lead to a simple and rigorous analytical procedure. The self-propulsion velocity and Lighthill's swimming efficiency are calculated analytically. It appears that V-robot can swim in both directions of x-axis, and V-robot, aligned perpendicularly to the direction of self-swimming, is the most efficient one. In addition, such V-robot is faster and more efficient than a linear three-sphere micro-robot, the most studied one. V-robot has already been studied numerically by Earl *et.al.* (2007), but never analytically. Our approach is technically different from all previous methods employed in the studies of micro-robots (except Vladimirov (2012b)). The possibility to derive an explicit formula (2.10) for a V-shape three-sphere micro-robot shows its strength and analytical simplicity. The used version of the two-timing method has been developed in Vladimirov (2005), Vladimirov (2008), Vladimirov (2012a).

We consider a symmetric V-shape micro-robot (V-robot) consisting of three rigid spheres of radii R_ν, $\nu = 0, 1, 2$ ($R_1 = R_2$) connected by two arms of equal length l. The angle between the arms is 2φ. In plane cartesian coordinates the centers of the



spheres and the distances between them are

$$\boldsymbol{x}^{(\nu)} = (x_1^{(\nu)}, x_2^{(\nu)}) \equiv (x^{(\nu)}, y^{(\nu)}), \quad \boldsymbol{r}^{(\mu\nu)} \equiv \boldsymbol{x}^{(\nu)} - \boldsymbol{x}^{(\mu)}$$

We accept the notations (see the figure):

$$\begin{aligned}
&\boldsymbol{x}^{(0)} = (x^{(0)}, y^{(0)}) = (X, 0), \quad \boldsymbol{x}^{(1)} = (x^{(1)}, y^{(1)}) = (x, y) = (X+\xi, y), \\
&\boldsymbol{x}^{(2)} = (x^{(2)}, y^{(2)}) = (x, -y) = (X+\xi, -y); \\
&\boldsymbol{r}^{(01)} \equiv (\xi, y), \quad \boldsymbol{r}^{(02)} = (\xi, -y), \quad \boldsymbol{r}^{(21)} = (2y, 0); \\
&|\boldsymbol{r}^{(01)}| = |\boldsymbol{r}^{(02)}| = l, \quad |\boldsymbol{r}^{(21)}| = 2y = 2l\sin\varphi, \quad (\xi, y) = l(\cos\varphi, \sin\varphi); \\
&\boldsymbol{n}^{(01)} = (\cos\varphi, \sin\varphi), \quad \boldsymbol{n}^{(02)} = (\cos\varphi, -\sin\varphi), \quad \boldsymbol{n}^{(21)} = (0, 1)
\end{aligned} \tag{1.1}$$

where $\boldsymbol{n}^{(\mu\nu)}$ are unit vectors along $\boldsymbol{r}^{(\mu\nu)}$. We use latin subscripts $(i, k = 1, 2)$ for cartesian components of vectors and tensors, subscript $\alpha = 1, 2, 3$ – for generalised coordinates, and subscripts (or superscripts) $\mu, \nu = 0, 1, 2$ to identify the spheres. V-robot represents a mechanical system described by three scalar parameters (generalised coordinates)

$$\boldsymbol{q} = (q_1, q_2, q_3) \equiv (X, l, \varphi) \tag{1.2}$$

V-robot moves due to the prescribed motion of the arms

$$l = L + \varepsilon\widetilde{l}(\tau), \quad \varphi = \Phi + \varepsilon\widetilde{\varphi}(\tau); \quad \tau \equiv \omega t, \ \omega = \text{const}, \quad \varepsilon = \text{const} \tag{1.3}$$

where the functions $\widetilde{l}(\tau)$ and $\widetilde{\varphi}(\tau)$ are 2π-periodic with zero average values, while L and Φ are constants. The spheres experience external friction forces $\boldsymbol{F}^{(\nu)} = (F_1^{(\nu)}, F_2^{(\nu)})$; however, the arms are chosen to be so thin in comparison with any R_ν, that their interaction with a fluid is negligible.

The considered problem contains three characteristic lengths: the length of arms L_{char}, the radius of spheres R_{char}, and amplitude of the arm's oscillations a_{char}. The characteristic time-scale is $T_{\text{char}} \equiv 1/\omega$ and the characteristic force is F_{char}. We have chosen

$$L_{\text{char}} \equiv L, \quad R_{\text{char}} \equiv (R_0 + 2R_1)/3, \ a_{\text{char}} \equiv \varepsilon L, \ F_{\text{char}} \equiv 6\pi\eta R_{\text{char}} L_{\text{char}}/T_{\text{char}} \tag{1.4}$$

where η is viscosity of a fluid. Two small parameters are

$$\varepsilon \equiv a_{\text{char}}/L \ll 1, \quad \delta \equiv 3R_{\text{char}}/(2L) \ll 1 \tag{1.5}$$

The dimensionless (asteriated) variables are chosen as

$$\boldsymbol{x} = L\boldsymbol{x}^*, \quad R_i = R_{\text{char}}R_i^*, \quad t = T_{\text{char}}t^*, \quad f_i = F_{\text{char}}f_i^*; \tag{1.6}$$

Below we use only dimensionless variables, however all asterisks are omitted.

Generalized coordinates $\boldsymbol{q} = \boldsymbol{q}(t)$ (1.2) determine the motion of V-robot. It can be described by the Lagrangian function $\mathcal{L} = \mathcal{L}(\boldsymbol{q}, \boldsymbol{q}_t)$, which includes the constraints (1.3)

$$\mathcal{L}(\boldsymbol{q}, \boldsymbol{q}_t) = \mathcal{K} + f(l - 1 - \varepsilon\widetilde{l}) + g(\varphi - \Phi - \varepsilon\widetilde{\varphi}) \tag{1.7}$$

where subscript t stands for d/dt, f and g are Lagrangian multipliers, and $\mathcal{K}$ is kinetic energy of a robot; notice that in dimensionless variables $L = 1$. The Lagrange equations are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial q_{\alpha t}} - \frac{\partial \mathcal{L}}{\partial q_\alpha} = Q_\alpha, \quad Q_\alpha = \sum_{\nu=0}^{2}\sum_{j=1}^{2} F_j^{(\nu)}\frac{\partial x_j^{(\nu)}}{\partial q_\alpha} \tag{1.8}$$

where $\boldsymbol{Q} = (Q_1, Q_2, Q_3)$ is the generalized external force, exerted by a fluid on V-robot.

The fluid flow past V-robot is described by the Stokes equations. In consistent approximation masses of spheres and arms are negligible, then $\mathcal{K} \equiv 0$. Hence (1.7),(1.8) produce the system of equations

$$Q_1 = 0, \quad Q_2 + f = 0, \quad Q_3 + g = 0 \tag{1.9}$$

The calculations of Q_α (1.8) with the use of (1.1) yield

$$\begin{aligned}
&Q_1 = F_1^{(0)} + F^+, \quad Q_2 = F^+\cos\varphi + F^-\sin\varphi, \\
&Q_3 = -lF^+\sin\varphi + lF^-\cos\varphi; \\
&F^+ \equiv F_1^{(1)} + F_1^{(2)}, \quad F^- \equiv F_2^{(1)} - F_2^{(2)}
\end{aligned} \tag{1.10}$$

The substitution of (1.10) into (1.9) and simple transformations yield

$$\begin{aligned}
&F_1^{(0)} + F^+ = 0, \quad F^+ = \lambda, \quad F^- = \sigma; \\
&\lambda \equiv -f\cos\varphi + (g/l)\sin\varphi, \quad \sigma \equiv -(g/l)\cos\varphi - f\sin\varphi,
\end{aligned} \tag{1.11}$$

The explicit expressions for $\boldsymbol{F}^{(\nu)}$ are

$$\begin{aligned}
&\boldsymbol{F}^{(\nu)} \simeq -R_\nu\boldsymbol{x}_t^{(\nu)} + \delta\sum_{\mu\neq\nu} R_\mu R_\nu \mathbb{S}^{(\nu\mu)}\boldsymbol{x}_t^{(\mu)} \\
&\mathbb{S}^{(\mu\nu)} = S_{ik}^{(\mu\nu)} \equiv \frac{1}{|\boldsymbol{r}^{(\mu\nu)}|}(\delta_{ik} + n_i^{(\mu\nu)}n_k^{(\mu\nu)})
\end{aligned} \tag{1.12}$$

Each force $\boldsymbol{F}^{(\nu)}$ represents the first approximation for the Stokes friction force exerted on a sphere moving in the flow generated by the other two spheres. To construct (1.12) we use a classical explicit formula for a fluid velocity past a moving sphere, see Lamb (1932), Landau & Lifshitz (1959), Moffatt (1996). The substitution of (1.12) into (1.11) yields:

$$\begin{aligned}
&R_0x_{1t}^{(0)} + R_1(x_{1t}^{(1)} + x_{1t}^{(2)}) - \delta A = 0 \\
&R_1(x_{1t}^{(1)} + x_{1t}^{(2)}) - \delta B = \lambda \\
&R_1(x_{2t}^{(1)} - x_{2t}^{(2)}) - \delta C = \sigma
\end{aligned} \tag{1.13}$$

where

$$A \equiv R_0 R_1 \left[S^{(01)}_{1k} x^{(1)}_{kt} + S^{(02)}_{1k} x^{(2)}_{kt} + (S^{(01)}_{1k} + S^{(02)}_{1k}) x^{(0)}_{kt}\right] + R_1^2 S^{(21)}_{1k} (x^{(1)}_{kt} + x^{(2)}_{kt})$$

$$S^{(01)}_{1k} = \frac{1}{l}\left(1 + \xi^2/l^2, \xi y/l^2\right), \; S^{(02)}_{1k} = \frac{1}{l}\left(1 + \xi^2/l^2, -\xi y/l^2\right), \; S^{(21)}_{1k} = \frac{1}{2y}(0, 1)$$

with summation convention over $k = 1, 2$. Functions B and C will not affect the calculation of the self-propulsion velocity with precision linear in δ, therefore it is sufficient to write $B = O(1)$ and $C = O(1)$. The equations (1.13),(1.3), (where definitions (1.1) should be taken into account) represent a system of five equations for five unknown functions of time: X, ξ, y, λ, and σ.

2. Two-timing method and asymptotic procedure

2.1. *Functions and notations*

The following *dimensionless* notations and definitions are in use:
(i) s and τ denote slow time and fast time; subscripts τ and s stand for related partial derivatives.
(ii) A dimensionless function, say $G = G(s, \tau)$, belongs to class $\mathcal{I}$ if $G = O(1)$ and all partial s-, and τ-derivatives of G (required for our consideration) are also $O(1)$. In this paper all functions belong to class $\mathcal{I}$, while all small parameters appear as explicit multipliers.
(iii) We consider only *periodic in τ functions* $\{G \in \mathcal{P} : \quad G(s, \tau) = G(s, \tau + 2\pi)\}$, where s-dependence is not specified. Hence all considered below functions belong to $\mathcal{P} \bigcap \mathcal{I}$.
(iv) For arbitrary $G \in \mathcal{P}$ the *averaging operation* is

$$\langle G \rangle \equiv \frac{1}{2\pi} \int_{\tau_0}^{\tau_0 + 2\pi} G(s, \tau)\, d\tau \equiv \overline{G}(s), \qquad \forall\ \tau_0 \tag{2.1}$$

(v) *The tilde-function* (or purely oscillating function) represents a special case of $\mathcal{P}$-function with zero average $\langle \widetilde{G} \rangle = 0$. The *bar-function* (or mean-function) $\overline{G} = \overline{G}(s)$ does not depend on τ. A unique decomposition $G = \overline{G} + \widetilde{G}$ is valid.

2.2. *Asymptotic procedure and successive approximations*

The introduction of fast time variable τ and slow time variable s represents a crucial step in our asymptotic procedure. We choose $\tau = t$ and $s = \varepsilon^2 t$. This choice can be justified by the same distinguished limit arguments as in Vladimirov (2012a). Here we present this choice without proof, however the most important part of this proof (that this choice leads to a valid asymptotic procedure) is exposed and exploited below. We use the chain rule

$$d/dt = \partial/\partial\tau + \varepsilon^2 \partial/\partial s \tag{2.2}$$

and then we accept (temporarily) that τ and s represent two independent variables. Unknown functions X, ξ, y, λ, and σ are taken as regular series

$$X(\tau, s) = X_0(\tau, s) + \varepsilon X_1(\tau, s) + \varepsilon^2 X_2(\tau, s) + \dots \tag{2.3}$$

and similar expressions for ξ, y, λ, and σ. In (2.3) we take $\widetilde{X}_0 \equiv 0$, which which express the basic property of considered solutions: long distances of self-swimming are caused by small oscillations. After the application of (2.2) to (2.3) we have

$$X_t = \varepsilon \widetilde{X}_{1\tau} + \varepsilon^2 (\widetilde{X}_{2\tau} + \overline{X}_{0s}) + O(\varepsilon^2) \tag{2.4}$$

and similar expressions for $\boldsymbol{x}^{(\nu)}$.

The successive approximations of equations (1.13) are:
Terms of order $\varepsilon^0 = 1$: $\lambda_0 \equiv 0$ and $\sigma_0 \equiv 0$;
Terms of order ε^1:

$$3\widetilde{X}_{1\tau} = -2R_1\widetilde{\xi}_{1\tau}, \quad \widetilde{X}_{1\tau} + \widetilde{\xi}_{1\tau} = \lambda_1/2, \quad \widetilde{y}_{1\tau} = \sigma_1/2 \tag{2.5}$$

Its average part gives $\overline{\lambda}_1 \equiv 0$ and $\overline{\sigma}_1 \equiv 0$, while its oscillating part leads to:

$$\begin{aligned} \widetilde{X}_1 &= -\frac{2R_1}{3}\widetilde{\xi}_1 = -\frac{2R_1}{3}(\widetilde{l}\cos\Phi - \widetilde{\varphi}\sin\Phi) \\ \widetilde{\lambda}_1 &= \frac{2R_0}{3}(\widetilde{l}_\tau\cos\Phi - \widetilde{\varphi}_\tau\sin\Phi) \\ \widetilde{\sigma}_1 &= 2(\widetilde{l}_\tau\sin\Phi + \widetilde{\varphi}_\tau\cos\Phi); \end{aligned} \tag{2.6}$$

We have used that in dimensionless variables (1.4),(1.6)

$$R_0 + 2R_1 = 3, \;\; R_0 = 3/(1+2\rho), \;\; R_1 = R_2 = 3\rho/(1+2\rho), \quad \rho \equiv R_1/R_0; \tag{2.7}$$

Terms of order ε^2: Here we consider only the first eqn.(1.13), which can be rewritten as

$$R_0X_t + 2R_1x_t = \delta R_1\left\{\frac{2R_0}{l}\left[(1+\xi^2/l^2)(X_t+x_t) + \frac{\xi y}{l^2}y_t\right] + \frac{R_1x_t}{y}\right\} \tag{2.8}$$

where $x = X + \xi$. Since we consider only linear in δ precision, then in the second approximation we should substitute (into the right hand side of (2.8)) the solutions from the first equality in (2.6). Further transformations yield

$$\begin{aligned} \overline{X}_{0s} &= -\frac{\delta R_1}{3}\left[\frac{2R_0(R_0-2R_1)}{3}\langle\widetilde{\xi}_1\widetilde{G}_{1\tau}\rangle + 2R_0\langle\widetilde{y}_1\widetilde{H}_{1\tau}\rangle + \frac{R_0R_1}{3}\langle\widetilde{\xi}_1\widetilde{K}_{1\tau}\rangle\right] \\ G &\equiv (1+\xi^2/l^2)/l, \quad H \equiv \xi y/l^3, \quad K \equiv 1/y \end{aligned} \tag{2.9}$$

It is instructive to write that in the original general notations (1.12) formulae (2.9),(2.7) can be presented as

$$\overline{X}_{0s} = -\frac{1}{3}\delta\sum_{k=1}^{2}\sum_{\nu=0}^{2}\sum_{\mu\neq\nu}R_\mu R_\nu\langle\widetilde{S}^{(\nu\mu)}_{1k\tau}\widetilde{x}^{(\mu)}_k\rangle$$

where both tilde-functions in the right hand side are from the first approximation in ε.

2.3. *Self-propulsion velocity*

The self-propulsion velocity is defined as $\overline{V}_0 \equiv \overline{X}_t \simeq \varepsilon^2\overline{X}_{0s}$. Expressions (2.9),(2.7) lead to the formula

$$\begin{aligned} \overline{V}_0 &= 2\varepsilon^2\delta\, U(\Phi,\rho)\langle\widetilde{l}\widetilde{\phi}_\tau\rangle, \\ U(\Phi,\rho) &\equiv \frac{3\rho}{2(1+2\rho)^3}\left[4\sin\Phi(1-6\rho\cos^2\Phi) + \frac{\rho}{\sin^2\Phi}\right] \end{aligned} \tag{2.10}$$

which represents the main result of the paper. Let us discuss it in detail:

(i) $\overline{V}_0$ is proportional to the correlation $\langle\widetilde{l}\widetilde{\phi}_\tau\rangle$; without any restriction of generality we consider only $\langle\widetilde{l}\widetilde{\phi}_\tau\rangle > 0$.

(ii) Further simplification can be achieved if we accept that the oscillations of arms are harmonic

$$\widetilde{l} = \cos(\tau+\theta_1), \quad \widetilde{\varphi} = \cos(\tau+\theta_2); \quad 2\langle\widetilde{l}\widetilde{\varphi}_\tau\rangle = \sin\theta, \quad \theta \equiv \theta_1 - \theta_2 \tag{2.11}$$

with constant phases θ_1, θ_2; $0 \le \theta \le \pi$. In this case the optimal stroke (providing the maximum of $\overline{V}_0$) is $\theta = \pi/2$, and $\max\langle \widetilde{l}\widetilde{\varphi}_\tau\rangle = 1/2$, when the self-propulsion velocity is:

$$\overline{V}_0 = \varepsilon^2 \delta\, U(\Phi, \rho) \tag{2.12}$$

(iii) $\overline{V}_0$ (2.10),(2.12) represents a function of two independent variables Φ and ρ for the domain $0 < \Phi < \pi$ and $0 < \rho < \infty$. Due to symmetry $\overline{V}_0(\pi/2 + \phi, \rho) = \overline{V}_0(\pi/2 - \phi, \rho)$ with $0 < \phi < \pi/2$, the backward-oriented (see the figure) V-robots with $\Phi = \pi/2 - \phi$ and forward-orientated V-robots with $\Phi = \pi/2 + \phi$ swim with the same velocity. Hence it is sufficient to study the domain

$$0 < \Phi < \pi/2, \quad 0 < \rho < \infty, \quad 0 < \theta < \pi \tag{2.13}$$

where the restriction $\theta < \pi$ appears due to $\langle \widetilde{l}\widetilde{\phi}_\tau\rangle > 0$.

(vi) A sharp singularity $\overline{V}_0 \to +\infty$ takes place for $\Phi \to 0$. However this limit does not have any physical meaning, since small values of Φ correspond to collision and overlapping of spheres R_1 and R_2. In order to avoid such collision and overlapping, one have to accept $\sin\Phi > 3\delta/2 > \delta R_1$. However, a stronger restriction is required in order to provide the validity of approximation for a velocity field in (1.12), where our basic assumption is: the radius of each sphere is much smaller than any distance between them; in particular, it means that $\sin\Phi \gg \delta R_1$. In practice one can take, say, $\sin\Phi > 5\delta$, which for $\delta = 0.1$ gives a 'secure' domain instead of (2.13):

$$\pi/6 < \Phi < \pi/2, \quad 0 < \rho < \infty, \quad 0 < \theta < \pi \tag{2.14}$$

More generally, the study of motion with small values of Φ requires the estimation of errors for the used approximation; the best way of doing it is computational, see Earl *et.al.* (2007). We do not consider this complex problem and restrict ourselves to the consideration of internal maxima of $\overline{V}_0$.

(v) $\overline{V}_0 \to 0$ when $R_0 \to 0$ (or $\rho \to \infty$), which corresponds to the well-known fact that a dumbbell with oscillating arm is not able to swim; also $\overline{V}_0 \to 0$ when $R_1 \to 0$ (or $\rho \to 0$), which represents a limiting case of one sphere without any oscillations.

(vi) A local maximum of $\overline{V}_0$ for any $\rho = \text{const}$ always takes place at $\Phi = \pi/2$, when

$$U(\pi/2, \rho) = 3\rho(4 + \rho)/[2(1 + 2\rho)^3] \tag{2.15}$$

Hence, a completely 'open' V-robot (with $2\Phi = \pi$) swims in positive direction (see the figure) with the maximal speed. Function $U(\pi/2, \rho)$ (2.19) is increasing linearly for small ρ; U reaches $\max U \simeq 0.47$ at $\rho \simeq 0.27$, and then U decreases to zero rapidly and monotonically. Hence we can write that

$$\max \overline{V}_0(\Phi, \rho, \theta) \simeq 0.47\epsilon^2\delta \quad \text{at} \quad \Phi = \pi/2, \quad \rho \simeq 0.27, \quad \theta = \pi/2 \tag{2.16}$$

It is interesting to compare this result with the result for a homogeneous linear three-sphere micro-swimmer, where $\max \overline{V}_0 \simeq 0.19\epsilon^2\delta$ (see Vladimirov (2012b)). The comparison shows that V-robot swims about 2.5 times faster than a linear micro-robot (when in both cases the strokes are harmonic and optimal). The fastest swimming of V-robot takes place when the spheres R_1 and R_2 are approximately 4 times smaller than R_0.

(vii) V-robot can swim in negative direction (see the figure) at $\Phi = \pi/4$, when the function $U(\pi/4, \rho)$ is non-monotonic and changes its sign: U increases linearly for small ρ, then U reaches $\max U \simeq 0.19$ at $\rho \simeq 0.13$, after that decreases such that $U = 0$ at $\rho = 0.43$. For $\rho > 0.43$ we have $U < 0$ and $\min U = U(1.75) \simeq -0.24$ which yields

$$\min \overline{V}_0(\pi/4, \rho, \pi/2) \simeq -0.24\,\epsilon^2\delta \quad \text{at} \quad \rho \simeq 2.10 \tag{2.17}$$

It follows from (2.10) that $\min \overline{V}_0$ in all domain (2.13) is

$$\min \overline{V}_0(\Phi, \rho, \theta) \simeq -0.26\,\epsilon^2\delta \quad \text{at} \quad \Phi = 0.69, \quad \rho \simeq 1.65, \quad \theta = \pi/2 \tag{2.18}$$

which is close to (2.17). Hence V-robot swims in negative direction if the angle between arms is close to $\pi/2$ (or $\Phi = \pi/4$). The maximal speed of this reverse swimming is approximately two times slower than the maximal speed in positive direction.

(viii) It is also of interest that for three equal spheres ($\rho = 1$) we have

$$\begin{aligned}
&\max \overline{V}_0(\Phi, 1, \theta) \simeq 0.28\epsilon^2\delta \quad \text{at} \quad \Phi = \pi/2, \quad \theta = \pi/2\\
&\min \overline{V}_0(\Phi, 1, \theta) \simeq -0.23\,\epsilon^2\delta \quad \text{at} \quad \Phi = 0.68, \quad \theta = \pi/2
\end{aligned}$$

which shows an essential reduction of the speed in positive direction.

(ix) The calculations at the boarder $\Phi = \pi/6$ of a 'secure' domain (2.14) show that $U(\pi/6, \rho)$ increases linearly for small ρ, reaches $\max U \simeq 0.13$ at $\rho \simeq 0.13$, then decreases such that $U(0.40) \simeq 0$ and $U < 0$ for $\rho > 0.40$; $\min U = U(1.65) \simeq -0.18$. These values show that $\Phi = \pi/6$ is still well away of a singularity at $\Phi \to 0$.

2.4. *Power and efficiency*

The power of V-robot is defined as

$$\mathcal{P} \equiv \sum_{\nu=0}^{2} \boldsymbol{f}^{(\nu)} \cdot \boldsymbol{x}_t^{(\nu)} \tag{2.19}$$

where $\boldsymbol{f}^{(\nu)}$ is the force, exerted by the arms on the ν-th sphere. The total force exerted on each sphere must be zero, hence $\boldsymbol{f}^{(\nu)} + \boldsymbol{F}^{(\nu)} = 0$ with the friction force $\boldsymbol{F}^{(\nu)}$ (1.12). Therefore the main term of (2.19) can be presented as

$$\mathcal{P} \simeq \sum_{\nu=0}^{2} R_\nu\, \boldsymbol{x}_t^{(\nu)} \cdot \boldsymbol{x}_t^{(\nu)} \simeq \varepsilon^2 \sum_{\nu=0}^{2} R_\nu\, \widetilde{\boldsymbol{x}}_{1\tau}^{(\nu)} \cdot \widetilde{\boldsymbol{x}}_{1\tau}^{(\nu)} \tag{2.20}$$

where we have used (2.4). Then the use of (2.5),(1.1),(2.1) yields

$$\overline{\mathcal{P}} \simeq \frac{3\varepsilon^2}{(1+2\rho)} \langle \widetilde{X}_{1\tau}^2 + 2\rho(\widetilde{x}_{1\tau}^2 + \widetilde{y}_{1\tau}^2)\rangle \tag{2.21}$$

and the use of (2.6) leads to

$$\overline{\mathcal{P}} \simeq \frac{6\varepsilon^2\rho}{(1+2\rho)^2} \langle \widetilde{l}_\tau^2 + \widetilde{\varphi}_\tau^2 + 2\rho[\widetilde{l}_\tau^2 \sin^2\Phi + \widetilde{\varphi}_\tau^2 \cos^2\Phi + \widetilde{l}_\tau \widetilde{\varphi}_\tau \sin(2\Phi)]\rangle \tag{2.22}$$

For harmonic oscillations (2.11) it gives

$$\overline{\mathcal{P}} \simeq \frac{6\varepsilon^2\rho}{(1+2\rho)^2} W, \quad W \equiv [1 + \rho\,(1 + \sin(2\Phi)\cos\theta)] \tag{2.23}$$

Another expression $\overline{\mathcal{P}}_s = 3\overline{V}_0^2$ represents the power, which is required to drag V-robot with velocity $\overline{V}_0$ in the absence of its oscillations when the main approximation for the dimensionless Stokes friction force is $-3\overline{V}_0$ (where the coefficient 3 represents the sum of all radii (2.7)). Lighthill's swimming efficiency (see Becker *et.al.* (2003)) is the ratio $\mathcal{E} \equiv \overline{\mathcal{P}}_s/\overline{\mathcal{P}}$. For harmonic oscillations (2.11) expressions (2.12),(2.23),(2.10) give

$$\mathcal{E} = \mathcal{E}(\Phi, \rho, \theta) \simeq \varepsilon^2\delta^2 \frac{9\rho}{8(1+2\rho)^4} \frac{U^2}{W} \sin^2\theta \tag{2.24}$$

The analysis and the computations show that

$$\max \mathcal{E} \simeq 0.89\varepsilon^2\delta^2, \quad \text{at} \quad \Phi = \pi/2, \quad \rho \simeq 0.15, \quad \theta = \pi/2 \tag{2.25}$$

One can see that $\max \mathcal{E}$ takes place at the same $\Phi = \pi/2$ and $\theta = \pi/2$, but for approximately twice smaller ρ than for $\max \overline{V}_0$ (2.16); hence the most efficient swimming takes place when the spheres R_1 and R_2 are approximately 7 times smaller than R_0. It is also interesting that for a linear three-sphere micro-robot $\max \mathcal{E} \simeq 0.18\varepsilon^2\delta^2$ (see Vladimirov (2012b)), which is five times lower than (2.25).

3. Discussion

(i) V-robot has already been studied numerically by Earl *et.al.* (2007), but never analytically. Quantitative comparison between our analytical results and computations of Earl *et.al.* (2007) is difficult, since these authors studied the large amplitudes of arms' oscillations and non-harmonic strokes; also the domain of the main parameters, they studied, is unclear from the text. At the same time the existence of both forward (2.16) and reverse (2.18) swimming provides qualitative agreement of our results with Earl *et.al.* (2007).

(ii) One can see that the magnitude of velocity in terms of small parameters $\overline{V}_0 = O(\varepsilon^2\delta)$ (2.10) is the same as the result by Golestanian & Ajdari (2008), Golestanian & Ajdari (2009) for a linear three-sphere robot and by Vladimirov (2012b) for a linear N-sphere robot. At the same time our choice of slow time $s = \varepsilon^2 t$ (2.2) agrees with classical studies of self-propulsion for low Reynolds numbers, see Taylor (1951), Blake (1971), Childress (1981), as well as the geometric studies of Shapere & Wilczek (1989).

(iii) In our examples, all arms move harmonically (2.11); it does not provide the maximum of $\overline{V}_0 \sim \langle \widetilde{l}\widetilde{\varphi}_\tau \rangle$ (2.10), see the relevant discussion in Vladimirov (2012b).The studies of non-periodic oscillations represent an interesting additional problem, see Golestanian & Ajdari (2009).

(iv) In our study we build an asymptotic procedure with two small parameters: $\varepsilon \to 0$ and $\delta \to 0$. Such a setting usually requires the consideration of different asymptotic paths on the plane (ε, δ) when, say $\delta = \delta(\varepsilon)$. In our case we can avoid such consideration, since small parameters appear (in the main order) as a product $\varepsilon^2\delta$.

(v) The mathematical justification of the presented results by the estimation of an error in the original equation can be performed similar to Vladimirov (2010), Vladimirov (2011). It is also possible to derive the higher approximations of $\overline{V}_0$, as it has been done by Vladimirov (2010), Vladimirov (2011) for different cases. The higher approximations can be useful for the studies of motion with $\overline{V}_0 \equiv 0$ (which are always possible for V-robots).

(vi) In order to compare the velocities of micro-robots and micro-organisms we use the dimensional variables, in which $\max \overline{V}_0^* \simeq 0.47\omega^* L^* \varepsilon^2\delta$; it shows that V-robot can move itself with the speed about 10 percent of its own size per second (we have taken $\varepsilon = \delta = 0.2$ and $\omega^* = 30s^{-1}$; the value of ω^* can be found in Pedley & Kessler (1987), Vladimirov *et.al.* (2004), Pedley (2009), Polin *et.al.* (2009). From these papers we also can see that this estimation of $\overline{V}_0^*$ is about 10 times lower than a similar value for natural micro-swimmers.

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